

On the structure of semigroups on L_p with a bounded H^∞ -calculus

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Notation: $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg(z)| < \varphi\}$.

Definition (Sectorial Operator)

$(A, D(A))$ densely defined operator with $\omega \in (0, \pi)$ such that

$$(S_\omega) \quad \sigma(A) \subset \overline{\Sigma_\omega} \quad \text{and} \quad \sup_{\lambda \notin \Sigma_{\omega+\varepsilon}} \|\lambda R(\lambda, A)\| < \infty \quad \forall \varepsilon > 0.$$

Then $\omega(A) := \inf\{\omega : (S_\omega) \text{ holds}\}$.

Definition (Analytic C_0 -semigroup)

Family of operators $(T(z))_{z \in \Sigma_\delta}$ ($\delta \in (0, \frac{\pi}{2})$) satisfying

- (i) $z \mapsto T(z)$ is analytic
- (ii) $T(z_1 + z_2) = T(z_1)T(z_2) \quad \forall z_1, z_2 \in \Sigma_\delta$
- (iii) $\lim_{\substack{z \rightarrow 0 \\ z \in \Sigma_{\delta'}}} T(z)x = x \quad \forall \delta' \in (0, \delta), \forall x \in X$

It is called bounded if $\sup_{z \in \Sigma_{\delta'}} \|T(z)\| < \infty$ for all $\delta' \in (0, \delta)$.

One has 1:1 correspondence

bounded analytic C_0 -semigroups $\leftrightarrow A$ sectorial with $\omega(A) < \frac{\pi}{2}$

At least formally $T(z) = e^{-zA}$.

Given $f \in H_0^\infty(\Sigma_\sigma) := \left\{ f : \Sigma_\sigma \rightarrow \mathbb{C} \text{ analytic} : |f(\lambda)| \leq \frac{|\lambda|^\varepsilon}{(1+|\lambda|)^{2\varepsilon}} \right\}$ define

$$f(A) := \int_{\partial\Sigma_{\sigma'}} f(\lambda) R(\lambda, A) d\lambda \quad (\omega(A) < \sigma' < \sigma).$$

Definition (Bounded H^∞ -calculus)

$(A, D(A))$ sectorial has bounded $H^\infty(\Sigma_\sigma)$ -calculus if for some $C \geq 0$

$$(H_\sigma) \quad \|f(A)\| \leq C \sup_{\lambda \in \Sigma_\sigma} |f(\lambda)| \quad \forall f \in H_0^\infty(\Sigma_\sigma).$$

Then $\omega_{H^\infty}(A) := \inf\{\sigma : (H_\sigma) \text{ holds}\}.$

Theorem (C. Le Merdy)

— $A \sim (T(z))_{z \in \Sigma}$ bounded analytic C_0 -semigroup on Hilbert space H .

Equivalent:

- (i) A has a bounded H^∞ -calculus
- (ii) there exists $S \in \mathcal{B}(H)$ invertible such that

$$\|S^{-1}T(t)S\| \leq 1 \quad \forall t \geq 0.$$

Put differently: contractive semigroups are generic for all semigroups with a bounded H^∞ -calculus.

Can this be generalized to L_p ($1 < p < \infty$)? In one direction, one has

Theorem (L. Weis)

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic C_0 -semigroup on L_p , positive and contractive on the real line.

Then A has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

Can all semigroups on L_p with a bounded H^∞ -calculus be obtained from such semigroups?

Theorem (S.F.)

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic C_0 -semigroup on $L_p(\Omega)$ ($1 < p < \infty$).

Equivalent:

- (i) A has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

Theorem (S.F.)

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic C_0 -semigroup on $L_p(\Omega)$ ($1 < p < \infty$).

Equivalent:

- (i) A has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.
- (ii) There exists a bounded holomorphic C_0 -semigroup $(R(z))_{z \in \tilde{\Sigma}}$ in some $L_p(\tilde{\Omega})$, positive and contractive on the real line with
 - $N \subset M \subset L_p(\tilde{\Omega})$ closed subspaces invariant under $(R(z))$
 - $S \in \mathcal{B}(L_p(\Omega), M/N)$ isomorphism

such that

$$T(z) = S^{-1} R_{M/N}(z) S \quad \forall z \in \tilde{\Sigma}.$$

Theorem (S.F. (Reminder))

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic C_0 -semigroup on $L_p(\Omega)$. Equivalent:

- (i) A has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.
- (ii) $T(z) = S^{-1}R_{M/N}(z)S \quad \forall z \in \tilde{\Sigma}$.

- On Hilbert spaces $\omega_{H^\infty}(A) < \frac{\pi}{2}$ holds automatically
- This seems to be open for L_p -spaces, but false for general subspaces of L_p -spaces (N.J. Kalton)
- (i) $\Rightarrow$ (ii) holds on every UMD-Banach lattice ($(R(z))$ lives on another UMD-Banach lattice)

Problem

Does the result hold without factorizing through a subspace-quotient as in the Hilbert space case?

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Problem

Does every positive contractive C_0 -semigroup on a UMD Banach lattice have a bounded H^∞ -calculus?

Main ideas of the proof (I):

- For $\alpha > 1$ give $H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}}+)$ a p -operator space structure as follows:

$$\begin{aligned} H^\infty(\Sigma_{\frac{\pi}{2\alpha}}+) &\hookrightarrow \mathcal{B}(L_p(\mathbb{R}; Y)) \\ f &\mapsto f(B_\alpha^{\frac{1}{\alpha}}), \end{aligned}$$

where $-B$ generates the shift semigroup $V(t)g(s) = g(s - t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued L_p -space Y .

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- p -complete boundedness of the functional calculus, e.g. mappings

$$\begin{aligned} \mathcal{B}(\ell_p^n(H^\infty(\Sigma))) \supset M_n(H^\infty) &\rightarrow M_n(\mathcal{B}(L_p(\mathbb{R}; Y))) \simeq \mathcal{B}(\ell_p^n(L_p(\mathbb{R}; Y))) \\ [f_{ij}] &\mapsto [f_{ij}(B)] \end{aligned}$$

are uniformly bounded in n .

Main ideas of the proof (II):

- A factorization theorem of G. Pisier yields a semigroup as asserted, except for strong continuity (ultraproduct construction).
- Reduce to the strongly continuous part.



Conclusions

Every bounded analytic C_0 -semigroup on $L_p(\Omega)$ with generator $-A$ satisfying $\omega_{H^\infty}(A) < \frac{\pi}{2}$ can be obtained

- from a bounded analytic C_0 -semigroup on $L_p(\tilde{\Omega})$, positive and contractive on the real line
- after passing to invariant subspace-quotients and similarity transforms

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Et bon appétit!